Fundamental Theorem of Calculus

Suppose f is continuous on $[a,b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

Ex 1: Find $g'(x)$ if $g(x) = \int_0^x \sqrt{1+t^2}\,dt$

$f(t) = \sqrt{1+t^2}$ is continuous on $[0,\infty)$

$\boxed{g'(x) = \sqrt{1+x^2}}$ FTC Part 1

Ex 2: Find $\frac{d}{dx}\int_1^{x^4} \sec t\,dt$

The x^4 doesn't look like the x in the theorem. Is that ok?!

Yes. Use u substitution for the x^4. Then, you have a function inside a function, so you'll need to use chain rule when you differentiate.

$u = x^4$

$du = 4x^3\,dx$

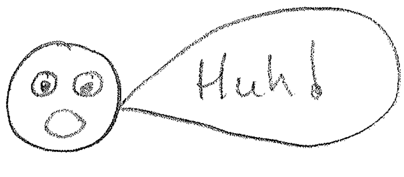
$\frac{du}{dx} = 4x^3$

$$= \underbrace{\frac{d}{du}\left(\int_1^u \sec t\,dt\right)}_{\text{Apply FTC 1}}\frac{du}{dx} \quad \text{Chain rule}$$

$$= \sec u \cdot \frac{du}{dx}$$

$$= \boxed{\sec(x^4)\,4x^3}$$

Ex 3: $\frac{d}{dx}\int_3^{x^2} \sin(t^2)\,dt = ?$



This is very confusing!
Is it ok for the dummy variable to be squared?!

Yes, the square is part of the function, not the dummy variable.

See it like this:

$$f(\square) = \sin(\square^2)$$

Whatever we input into the box, the function squares it, then takes its sine.

OK, so now this is just like Ex 2.

$$\frac{d}{dx}\int_3^{x^2} \sin(t^2)\,dt$$

$u = x^2$
$du = 2x\,dx \quad \frac{du}{dx} = 2x$

$$= \frac{d}{du}\int_3^{u} \sin(t^2)\,dt \cdot \frac{du}{dx}$$

$$= \sin(u^2)\cdot \frac{du}{dx}$$

$$= \sin((x^2)^2)\,2x$$

$$= \boxed{2x \sin x^4}$$

But I still don't really get FTC 1.
Why does the lower bound and the $\int$ disappear?
Is this just magic?

So basically, FTC 1 says that the derivative of the integral of a function is the function. It's kind of like saying "subtracting 2 from a number after adding 2 to a number just gets you the number".

FTC 2 says that when you take a definite integral, any antiderivative will do and you only need to look at the endpoints.

Since many people find FTC 2 easier to understand (we can draw it with graphs), let's see if FTC 2 can help us understand why the lower bound a disappears.

Back to Ex 3:

$$\frac{d}{dx}\int_3^{x^2} \sin(t^2)\,dt = ?$$

$f(t) = \sin(t^2)$

Let's substitute $f(t)$ to help us focus on the big picture.

$$\int_3^{x^2} f(t)\,dt$$

$$= F(x^2) - F(3)$$

FTC 2

$$\frac{d}{dx}\left[F(x^2) - F(3)\right]$$

$$= \frac{d}{dx}\left[F(x^2)\right] - \frac{d}{dx}\left[F(3)\right]$$

$F(3)$ is just a number, so $\frac{d}{dx}\left[F(3)\right] = 0$!

0 Bye bye!

$$= \frac{d}{dx}\left[F(x^2)\right] - \cancel{\frac{d}{dx}\left[F(3)\right]}$$

Now, we have the derivative of a function inside a function.

Outer function = F

Inner function = x^2

So, we'll need to use chain rule.

Think $\frac{d}{dx}\left[F(\square^2)\right]$

$$= f(x^2)\cdot 2x$$

$$= \sin(x^4)\cdot 2x$$

$$= \boxed{2x\sin(x^4)}$$

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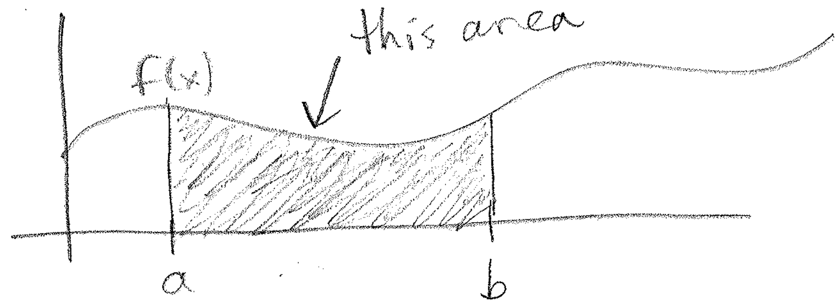
* FTC 2 help.

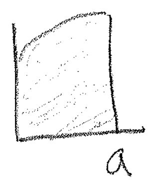
From your work with Riemann sums and integrals, you know that

$$\int_a^b f(x)\,dx =$$



Well, one valid $F(x) =$

$\int_0^x f(x)\,dx$ so $F(a) =$



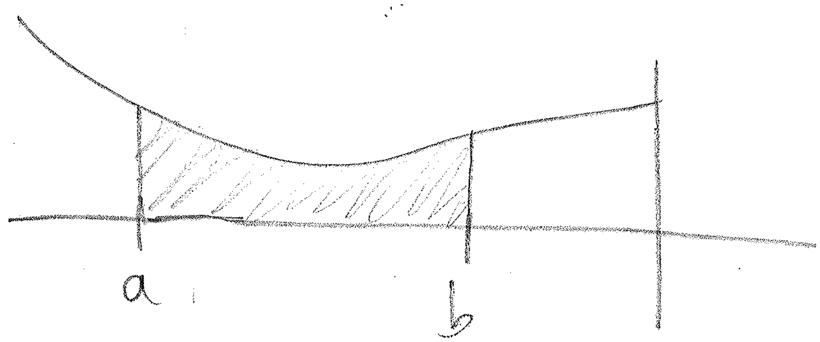
and $F(b) =$

So $F(b) - F(a)$ gets you the area from a to b.

If a and b are negative, this still works because

$$\int_0^{-2} f(x) = -\int_{-2}^{0} f(x)dx$$

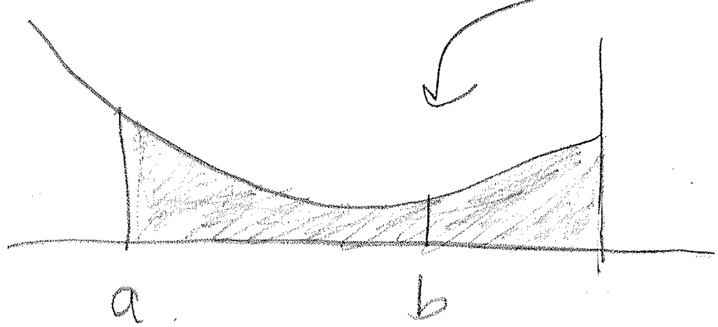
So if you have



$$= \int_a^b f(x)dx$$

$$= F(b) - F(a)$$

$$= -\int_b^0 f(x)dx + \int_a^0 f(x)dx$$



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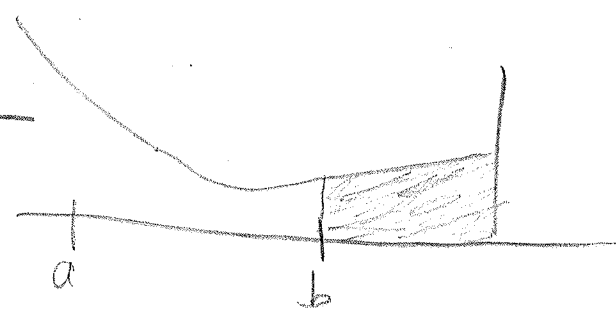

All other valid F's for this given function will be our F(x) + or − a constant. (That's why you add +C whenever you integrate). The +C's will cancel when you subtract: $F(b)+\cancel{C}-[F(a)+\cancel{C}]$, so that's why ANY antiderivative of f will work.